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School of Engineering

Finite Element Analysis
Basis of FEM

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EG3111 – Finite Element Analysis and Design

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1b. Introduction to the basis of FEM: the Weighted Residual Method

Weighted residual methods generate approximate solutions to PDEs.

This is the basis of the general FEM.

As an example, consider the ordinary differential equation

$$\frac{du}{dx} + u = 0 \qquad \text{for } 0 \leq x \geq 1$$

Subject to boundary condition

$$u(0) = 1$$

The exact solution is

$$u(x) = e^{-x}$$

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1b. One Degree of Freedom (DOF) Approximation

- We will use two different weighted residual methods to generate approximations to the exact solution:
  (i) Least squares error method
  (ii) Galerkin method
- **Shape function** – propose a form for the approximate solution…
- The simplest approximation to $u(x)$ is a straight line

$$\tilde{u}(x) = 1 + cx$$

Approximate solution for $u(x)$

To satisfy boundary condition

Constant c is to be determined to give best approximation

- The unknown variable c is called the **degree of freedom** (DOF) of the system**.**

1b. Residual error

- As an example, consider the ordinary differential equation

$$\frac{du}{dx} + u = 0 \qquad \text{for } 0 \leq x \geq 1$$

Subject to boundary condition

$$u(0) = 1$$

- **Shape function** – propose a form for the approximate solution…
- The simplest approximation to $u(x)$ is a straight line

$$\tilde{u}(x) = 1 + cx$$

- Define the residual error $R(x)$ to be the difference between the exact value of the PDE (zero in this case) and the approximated value. In this example

$$R(x) = \left(\frac{d\tilde{u}}{dx} + \tilde{u}\right) - \left(\frac{du}{dx} + u\right) = \left(\frac{d\tilde{u}}{dx} + \tilde{u}\right)$$

- If the approximation is correct $(\tilde{u} = u)$ then the error $R(x) = 0$.
- For the 1 DOF approximation function

$$R(x) = \frac{d}{dx}(1 + cx) + (1 + cx) = c + 1 + cx$$

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1b. (i) Least Squares Method

- To find the best value of c minimise the sum of all the errors squared R^2
- Total error

$$I = \frac{1}{2}\int_0^1 R^2 \, dx$$

- Minimum error occurs when

$$\frac{dI}{dc} = 0$$

with respect to the DOF c such that

$$\frac{dI}{dc} = \frac{1}{2}\int_0^1 \frac{d(R^2)}{dc} \, dx$$

$$\frac{dI}{dc} = \frac{1}{2}\int_0^1 \frac{2RdR}{dc} dx$$

$$\frac{dI}{dc} = \int_0^1 \frac{dR}{dc} R dx = 0 \qquad \text{Eq 1.1}$$

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1b. (i) Least Squares Method (continued)

- In this example

$$\frac{dR}{dc} = 1 + x$$

$$\frac{dI}{dc} = \int_0^1 (1 + c + cx)(1 + x)dx = 0$$

$$\frac{dI}{dc} = c\int_0^1 (1 + x)^2\, dx + (1 + x)dx = 0$$

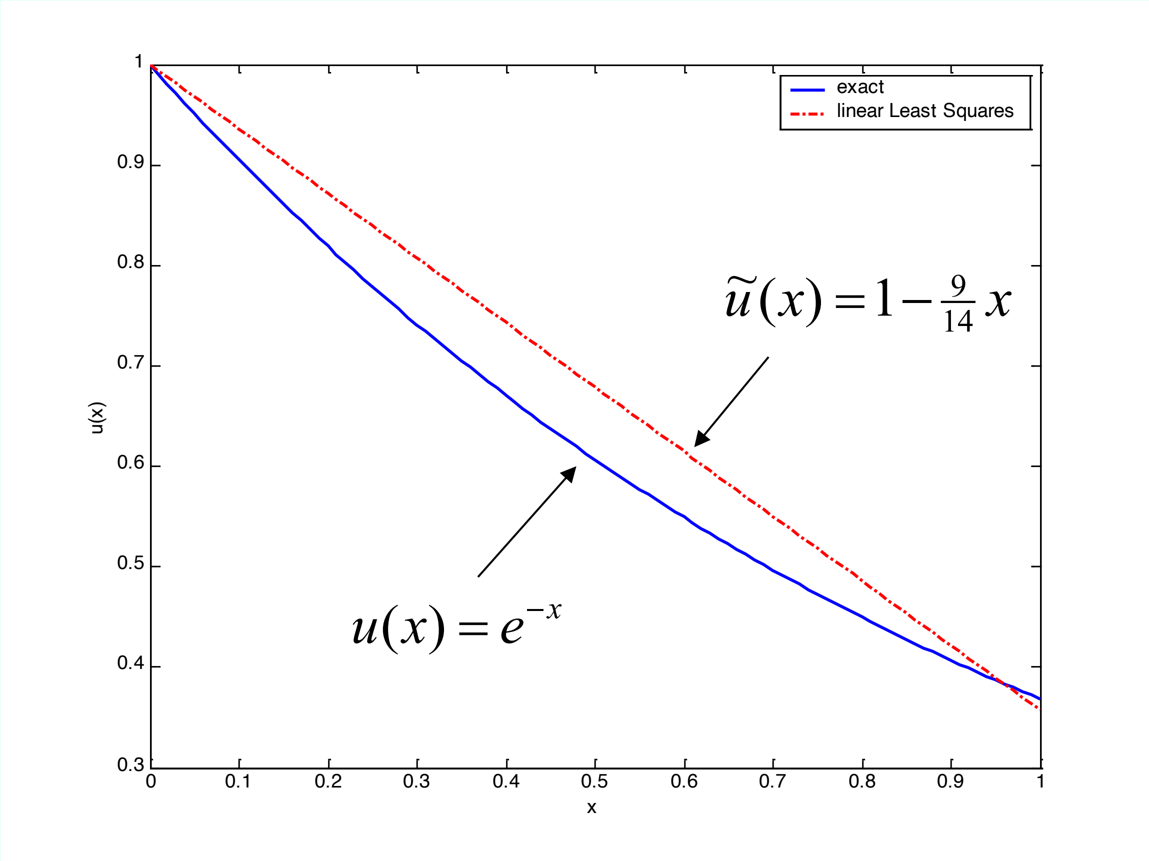
$$\frac{dI}{dc} = c\left[\frac{(1 + x)^3}{3}\right]_0^1 + \left[\frac{(1 + x)^2}{2}\right]_0^1$$

$$\frac{dI}{dc} = c\left(\frac{8}{3} - \frac{1}{3}\right) + \left(2 - \frac{1}{2}\right)$$

$$\frac{dI}{dc} = \frac{7c}{3} + \frac{3}{2} = 0$$

$$c = -\frac{3}{3} \times \frac{3}{7} = -\frac{9}{14}$$

1b. (i) Least Squares Method
Linear (1 DOF) approximation vs exact solution



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1b. (ii) Galerkin Method

- The Galerkin method is the basis of the FEM and is a generalisation of Eq 1.1

$$\int_0^1 w(x) R dx = 0$$

Where $w(x)$ is a weighting function

- For the least squares method

$$w(x) = \frac{dR}{dc}$$

- For the Galerkin method

$$w(x) = \frac{d\tilde{u}}{dc}$$

1b. (ii) Galerkin Method (continued)

- In this example, here we write condition for optimal degree of freedom DOF as

$$\int_0^1 \frac{d\tilde{u}}{dc} R dx = 0$$

- We replace ($\frac{dR}{dc}$ by $\frac{d\tilde{u}}{dc}$)
- We know

$$\tilde{u}(x) = 1 + cx$$

- Gives

$$\frac{d\tilde{u}}{dc} = x$$

$$\frac{d\tilde{u}}{dc} = \int_0^1 x(1 + c + cx)dx = 0$$

$$\frac{d\tilde{u}}{dc} = c\int_0^1 x(1 + x)dx + \int_0^1 x\, dx = 0$$

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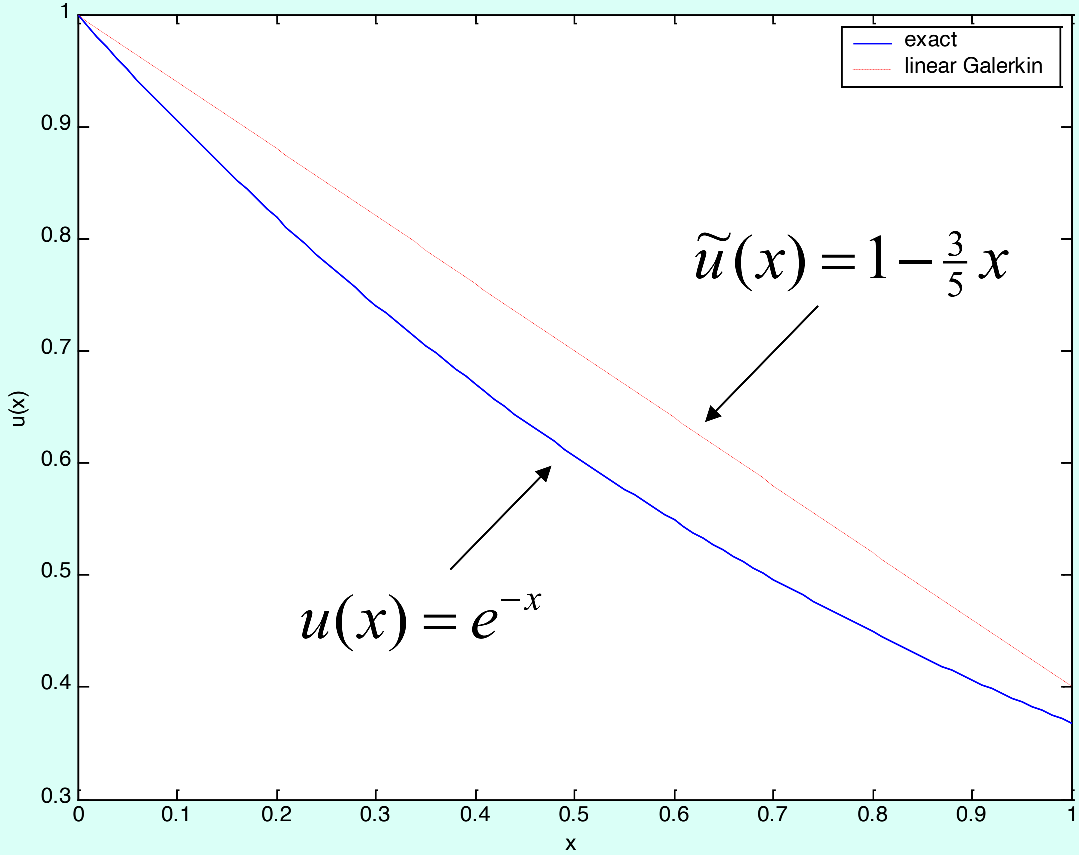
1b. (ii) Galerkin Method (continued)

$$\frac{d\tilde{u}}{dc} = c\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\frac{d\tilde{u}}{dc} = \frac{5c}{6} + \frac{1}{2} = 0$$

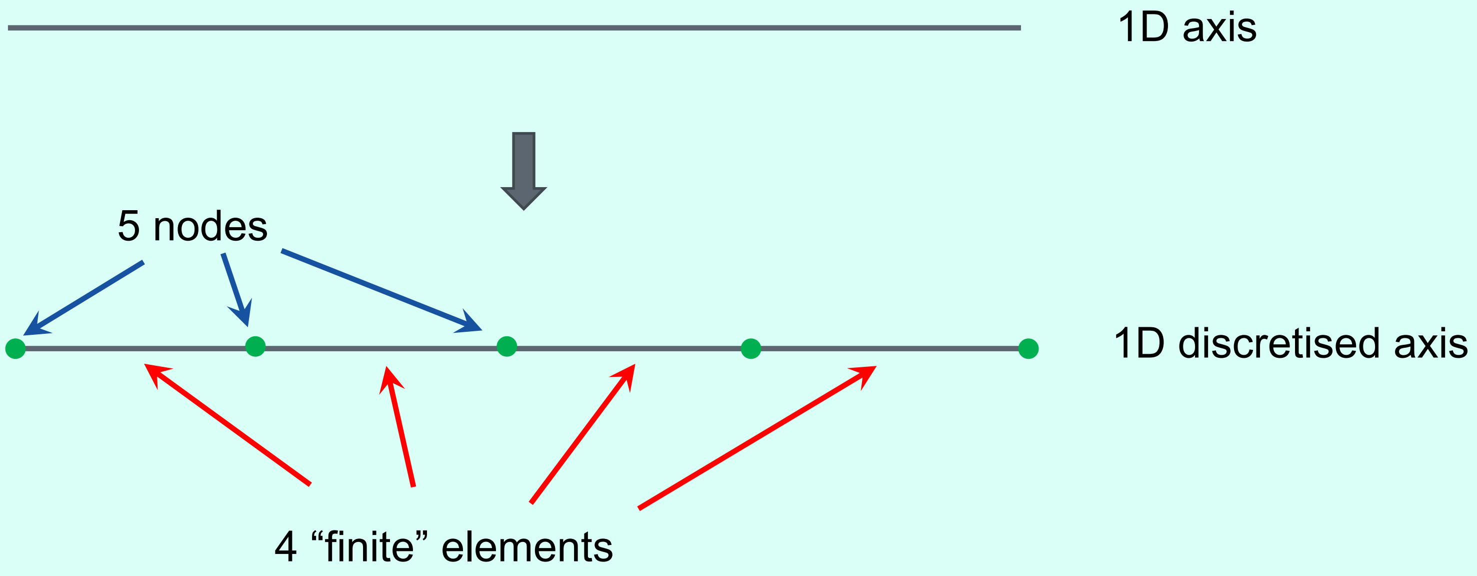
$$c = -\frac{1}{2} \times \frac{6}{5} = -\frac{3}{15}$$

1b. (ii) Galerkin
Linear (1 DOF) approximation vs exact solution



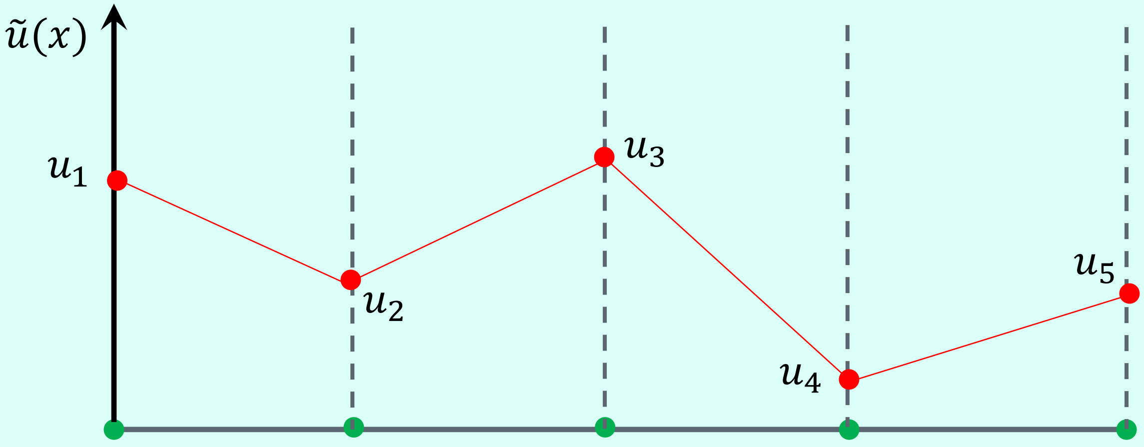
1b. Discretisation, shape functions, nodes and elements

- In general, the FEM discretises a body into elements



1b. Linear shape functions

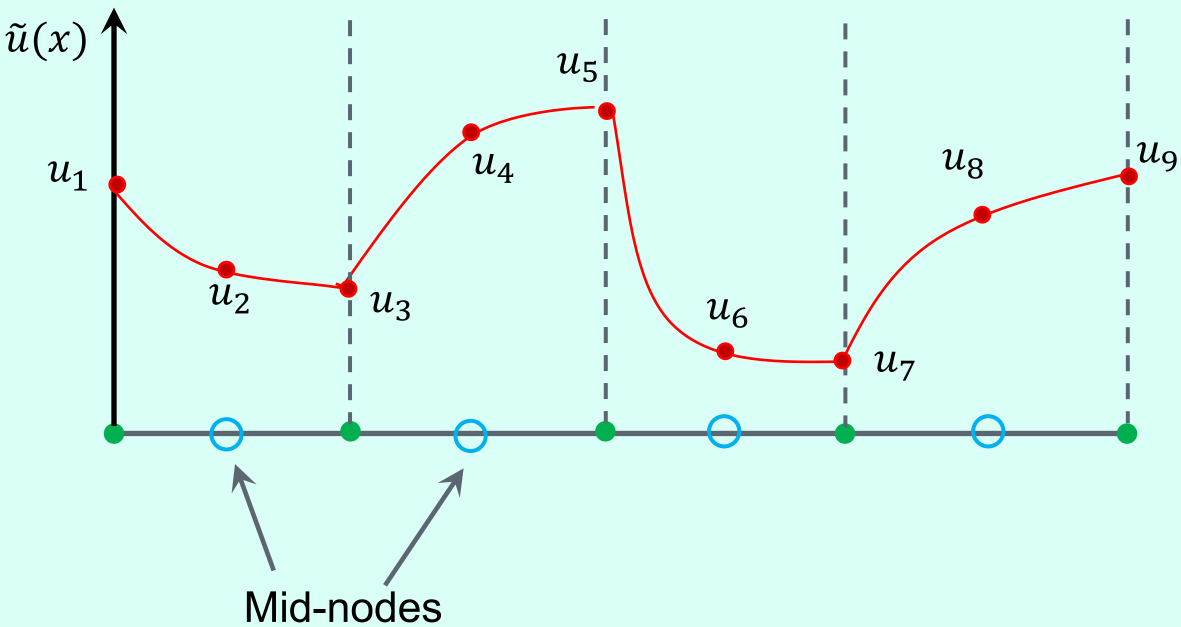
- The degrees of freedom (DOF) are the values at the nodes ($u_1, \dots, u_5$) to ensure continuity of the approximation function $\tilde{u}(x)$ between elements



5 DOF

1b. Quadratic shape functions

Quadratics need three values per element to define them, so introduce another DOF at mid-nodes



COMSOL has:

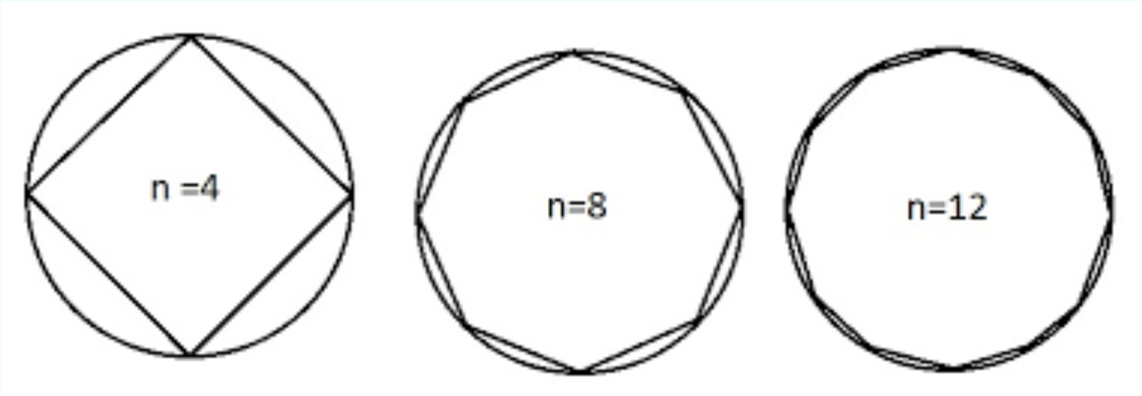
- Linear
- Quadratic
- Cubic
- Quartic
- Quintic

9 DOF

1b. Discretisation and accuracy

See solutions to Exercise sheet #1

- Increasing the number of DOF increases the accuracy of the solution
- Two ways to increase the DOF:
  - Increase the number of elements, i.e. decrease the size of the elements
  - Increase the order of the shape function, i.e. go from linear (1 DOF per element) to quadratic (2 DOF per element).



Approximating a circle with linear shape functions

Increasing DOF

Example #1: quadratic shape function

Use Galerkins method to find the approximate solution when the shape function is a second order polynomial (i.e. a quadratic curve rather than a linear curve).

$$\tilde{u}(x) = 1 + c_1 x + c_2 x^2$$

As there are now **2 DOF** (c_1 and c_2) there are also two simultaneous equations to solve

$$\int_0^1 \frac{\partial \tilde{u}}{\partial c_1} R dx = 0$$

$$\int_0^1 \frac{\partial \tilde{u}}{\partial c_2} R dx = 0$$

Plot the result against the exact solution for comparison

Exercise sheet #1

Example #2: 2 piecewise linear shape functions

Use Galerkins method to find the approximate solution when the shape function is two straight lines such that

$$\tilde{u}(x) = \begin{cases} 1 + 2(u_1 - 1)x & for\ 0 \leq x \leq \frac{1}{2} \\ (2u_1 - u_2) + 2(u_2 - u_1)x & for\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

There are now **two DOF** (u_1 and u_2) and hence two simultaneous equations to solve

$$\int_0^1 \frac{\partial \tilde{u}}{\partial u_1} R dx = 0$$

$$\int_0^1 \frac{\partial \tilde{u}}{\partial u_2} R dx = 0$$

Exercise sheet #1

Plot the result against the exact solution for comparison

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Next section…
(2) FEM and Elasticity Theory